TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

Financial & Actuarial Mathematics

Aggregation of Integer-Valued Risks with Copula-Induced Dependency Structure

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The multivariate distribution function F of a random vector $(X_1, \dots, X_d)$ contains two kinds of information:

- the univariate marginal distributions $F_1, \dots, F_d$ and
- the dependency structure among the components.

Definition

Let $d \in \mathbb{N}, d \geq 2$ and let $(U_1, \dots, U_d)$ denote a random vector on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that the random variable U_i is standard uniformly distributed for all $i = 1, \dots, d$.
A d-dimensional copula C is a multivariate distribution function on the d-dimensional unit cube with univariate standard uniform marginals,

$$\begin{aligned} C : [0,1]^d &\to [0,1] \\ (u_1, \dots, u_d) &\mapsto \mathbb{P}[U_1 \leq u_1, \dots, U_d \leq u_d]. \end{aligned}$$

Any dependency structure can be described using a copula

Sklar's theorem

Let F denote a multivariate distribution function with univariate marginals $F_1, \dots, F_d$. Then there exists a d-dimensional copula $C : [0,1]^d \to [0,1]$, such that for all $x_1, \dots, x_d \in \mathbb{R}$ it holds that

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)). \tag{1}$$

If $F_1, \dots, F_d$ are continuous, then C is unique. Conversely, if C is a copula and $F_1, \dots, F_d$ are univariate distribution functions, then the function F defined via (1) is a d-dimensional distribution function with one-dimensional margins $F_1, \dots, F_d$.

Restricted uniqueness of copulas in the case of non-continuous margins

Sklar's theorem states that for continuous marginal distributions the copula is unique on $[0,1]^d$.

$\Rightarrow$ **This is no longer true for discrete or mixed margins!**

Theorem

Let F denote the multivariate distribution function of a random vector $X = (X_1, \dots, X_d)$ on $\mathbb{R}^d$ with univariate marginals $F_1, \dots, F_d$. Then a copula C of X is uniquely determined on $\mathrm{Ran}(F_1) \times \cdots \times \mathrm{Ran}(F_d)$.

Extreme example: Consider Bernoulli-distributed random variables $X_1 \sim \mathrm{Bern}(p_1)$, $X_2 \sim \mathrm{Bern}(p_2)$ with $p_1, p_2 \in (0,1)$. Then a copula of (X_1, X_2) is only uniquely determined in the point $(1-p_1, 1-p_2)$.

Fundamental copulas

Independence copula

The independence copula $\Pi : [0,1]^d \to [0,1]$ is given by

$$\Pi(u_1, \dots, u_d) := \prod_{i=1}^{d} u_i, \quad u_1, \dots, u_d \in [0,1].$$

Comonotonicity copula

The comonotonicity copula $M : [0,1]^d \to [0,1]$ is given by

$$M(u_1, \dots, u_d) := \min\{u_1, \dots, u_d\}, \quad u_1, \dots, u_d \in [0,1].$$

Countermonotonicity copula (only in dimension $d = 2$)

The countermonotonicity copula $W : [0,1]^2 \to [0,1]$ is given by

$$W(u_1, u_2) := \max\{u_1 + u_2 - 1, 0\}, \quad u_1, u_2 \in [0,1].$$

Gaussian copulas

Definition

Let Φ denote the distribution function of a univariate standard normal distribution and let Φ_P^d be the distribution function of a d-variate normal distribution with correlation matrix P and mean 0. Then for $u_1, \ldots, u_d \in [0,1]$ the d-dimensional Gaussian copula $C_P^{\mathrm{Ga}} : [0,1]^d \to [0,1]$ is given as

$$C_P^{\mathrm{Ga}}(u_1, \ldots, u_d) := \Phi_P^d(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)).$$

- If $P = I_d$, where I_d denotes the d-dimensional identity matrix, then $C_P^{\mathrm{Ga}} \equiv \Pi$.
- If $P = J_d$, where J_d denotes a $d \times d$ matrix consisting entirely of ones, then $C_P^{\mathrm{Ga}} \equiv M$.
- If $d = 2$ and $P = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, then $C_P^{\mathrm{Ga}} \equiv W$.

t-copulas

Definition

Let t_ν be the distribution function of a univariate standard t-distribution with ν degrees of freedom, $\nu > 0$. By $t_{\nu,P}$ we denote the multivariate distribution function of a d-variate t-distribution with dispersion matrix P and $\nu > 0$ degrees of freedom. Then for $u_1, \ldots, u_d$ the d-dimensional t-copula $C^t_{\nu,P} : [0,1]^d \to [0,1]$ is given as

$$C^t_{\nu,P}(u_1, \ldots, u_d) := t_{\nu,P}(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)).$$

- **Caution!** If $P = I_d$, where I_d denotes the d-dimensional identity matrix, then $C^t_{\nu,P} \not\equiv \Pi$.
- If $P = J_d$, where J_d denotes a $d \times d$ matrix consisting entirely of ones, then $C^t_{\nu,P} \equiv M$.

Setting

- Arbitrary dimension $d \in \mathbb{N}_{\geq 2}$
- $\mathbb{N}_0$-valued random variables (risks) $X_1, \ldots, X_d$
- $X_i \sim F_i$ for $i = 1, \ldots, d$
- Dependency structure of the portfolio $(X_1, \ldots, X_d)$ is given by an arbitrary copula C
- **Aggregate portfolio loss** $S = \sum_{i=1}^{d} X_i$

Notational conventions

For $n \in \mathbb{N}_0$ we define

- $\mathcal{J}_n^d = \{j = (j_1, \ldots, j_d) \in \mathbb{N}_0^d : j_1 + \cdots + j_d \leq n\}$
- $\bar{\mathcal{J}}_n^d = \{j = (j_1, \ldots, j_d) \in \mathbb{N}_0^d : j_1 + \cdots + j_d = n\}$
- $p_n = \mathbb{P}[S = n]$
- $c_n = \sum_{j \in \bar{\mathcal{J}}_n^d} C(F_1(j_1), \ldots, F_d(j_d))$
- Convention: $p_n = c_n = 0$, if $n < 0$

Further,

- $\mathcal{I}^d = \{i = (i_1, \ldots, i_d) \in \{0, 1\}^d\}$
- $\mathrm{sign}(i) = (-1)^{\sum_{k=1}^d i_k}, \quad i \in \mathcal{I}^d$

Known results

Joint probability mass function of $(X_1, \dots, X_d)$

From the properties of copulas we know that for $j_1, \dots, j_d \in \mathbb{N}_0$

$$\mathbb{P}[X_1 = j_1, \dots, X_d = j_d] = \sum_{i \in \mathcal{I}^d} \mathrm{sign}(i) C(F_1(j_1 - i_1), \dots, F_d(j_d - i_d)).$$

Distribution of the sum S

$$\mathbb{P}[S \leq n] = \sum_{j \in \mathcal{J}_n^d} \mathbb{P}[X_1 = j_1, \dots, X_d = j_d], \quad n \in \mathbb{N}_0$$

Disadvantage of the **formula above: inefficient**, many summands
$\Rightarrow$ For $d = 6$ and $n = 100$ we sum up 109.177.903.744 terms!

Theorem (Distribution of the sum S)

Let $X_1, \ldots, X_d$ denote $\mathbb{N}_0$-valued random variables with univariate distribution functions $F_1, \ldots, F_d$, respectively. Then for $n \in \mathbb{N}_0$ and every copula C of the random vector $(X_1, \ldots, X_d)$ it holds that

$$\mathbb{P}[S \leq n] = \sum_{k=0}^{\min\{d-1,n\}} (-1)^k \binom{d-1}{k} c_{n-k}.$$

$\Rightarrow$ For $d = 6$ and $n = 100$ we **reduce** the **number of summands by** approximately **99.53%**

Recursion for the probability mass function of S

Remember:

- $\bar{\mathcal{J}}_n^d = \{j = (j_1, \dots, j_d) \in \mathbb{N}_0^d : j_1 + \dots + j_d = n\}$
- $p_n = \mathbb{P}[S = n]$
- $c_n = \sum_{j \in \bar{\mathcal{J}}_n^d} C(F_1(j_1), \dots, F_d(j_d))$
- Convention: $p_n = c_n = 0$, if $n < 0$

Recursion for the probability mass function of the sum S

Starting with $p_0 = c_0$, the following recursion formula applies:

$$p_n = c_n - \sum_{k=1}^{n} \binom{k+d-1}{d-1} p_{n-k}, \quad n \in \mathbb{N}.$$

Numerical speed-up

Disadvantages of the **recursion formula**:

- **Weak efficiency** – especially for sparse univariate marginal distributions or large dimensions d combined with large n
- **High precision** is **necessary**

Theorem (Probability mass function of the sum S)

Let $X_1, \ldots, X_d$ denote $\mathbb{N}_0$-valued random variables with univariate distribution functions $F_1, \ldots, F_d$, respectively. Then for all $n \in \mathbb{N}_0$ and every copula C of the random vector $(X_1, \ldots, X_d)$ it holds that

$$\mathbb{P}[S = n] = \sum_{k=0}^{\min\{d,n\}} (-1)^k \binom{d}{k} c_{n-k}.$$

Integration over copula densities

Definition

If the probability measure associated with a copula C is absolutely continuous with respect to the Lebesgue measure on $[0,1]^d$, then by Radon–Nikodým there exists an almost everywhere unique density $c : [0,1]^d \to [0,\infty)$ such that for $u_1,\dots,u_d \in [0,1]$:

$$C(u_1,\dots,u_d) = \int_0^{u_1} \cdots \int_0^{u_d} c(v_1,\dots,v_d)\, dv_d \dots dv_1.$$

Theorem (Probability mass function of the sum S)

For $n \in \mathbb{N}_0$ we have that

$$\mathbb{P}[S=n] = \sum_{j \in \bar{\mathcal{J}}_n^d} \int_{F_1(j_1-1)}^{F_1(j_1)} \cdots \int_{F_d(j_d-1)}^{F_d(j_d)} c(v_1,\dots,v_d)\, dv_d \dots dv_1.$$

Value-at-Risk and Expected Shortfall

Value-at-Risk

At a given confidence level $\alpha \in (0,1)$, the Value-at-Risk (VaR) of a random variable S is the smallest value $s \in \mathbb{R}$ where the distribution function F_S of S reaches or exceeds the value α for the first time:

$$\mathrm{VaR}_\alpha(S) := \inf\{s \in \mathbb{R} : F_S(s) \geq \alpha\}.$$

Expected Shortfall

For a random variable S with $\mathbb{E}[|S|] < \infty$ the Expected Shortfall (ES) at a confidence level $\alpha \in (0,1)$ is given as

$$\mathrm{ES}_\alpha(S) := \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_u(S)\, du.$$

Example

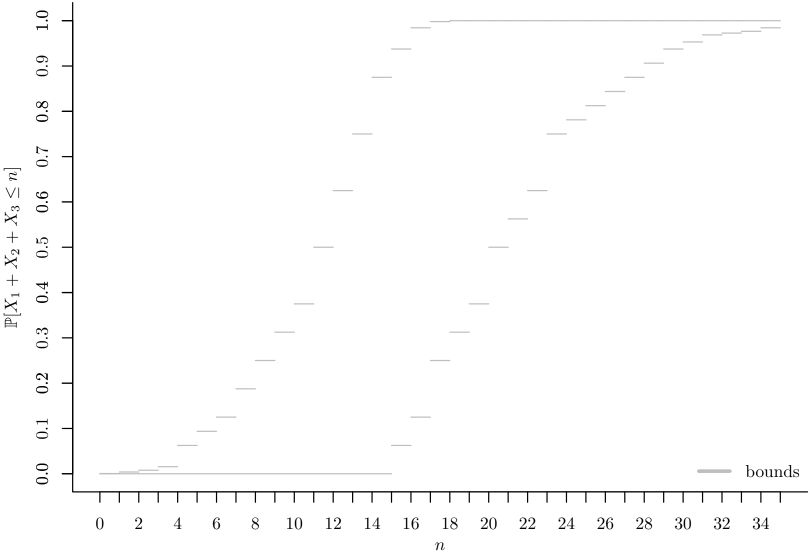
Let $d = 3$ and consider Poisson-distributed random variables

$$X_1 \sim \mathrm{Poi}(3),\ X_2 \sim \mathrm{Poi}(5) \text{ and } X_3 \sim \mathrm{Poi}(8).$$

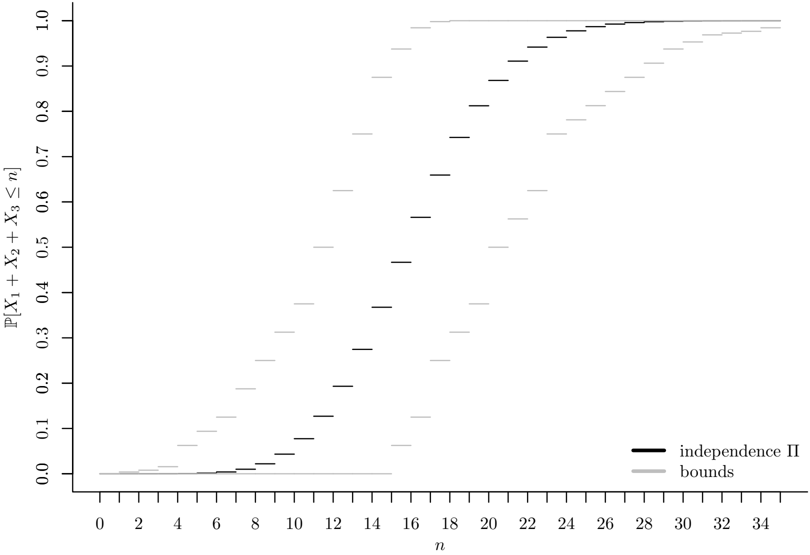
On the next slides we will present the distribution, probability mass function and risk measures for the sum $S = X_1 + X_2 + X_3$ under the following dependency scenarios:

- Fundamental copulas
- Gaussian copulas
- t-copulas with $\nu = 1$

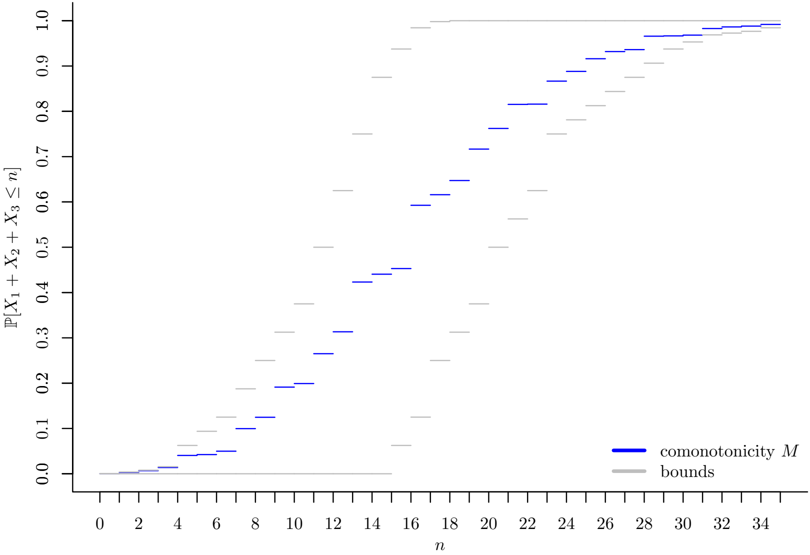
Fundamental copulas



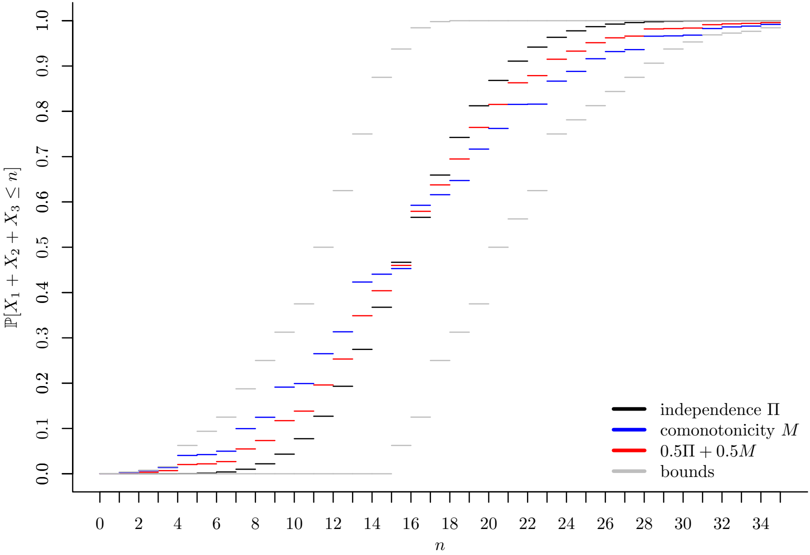
Fundamental copulas



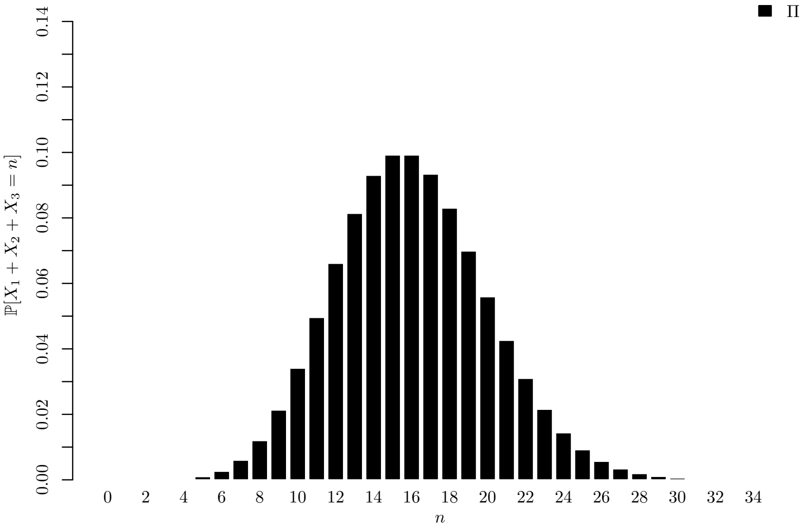
Fundamental copulas



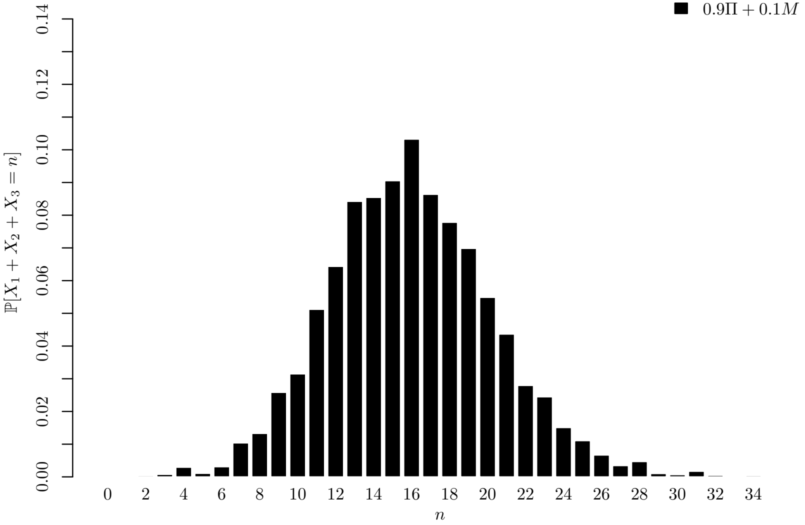
Fundamental copulas



Fundamental copulas



Fundamental copulas



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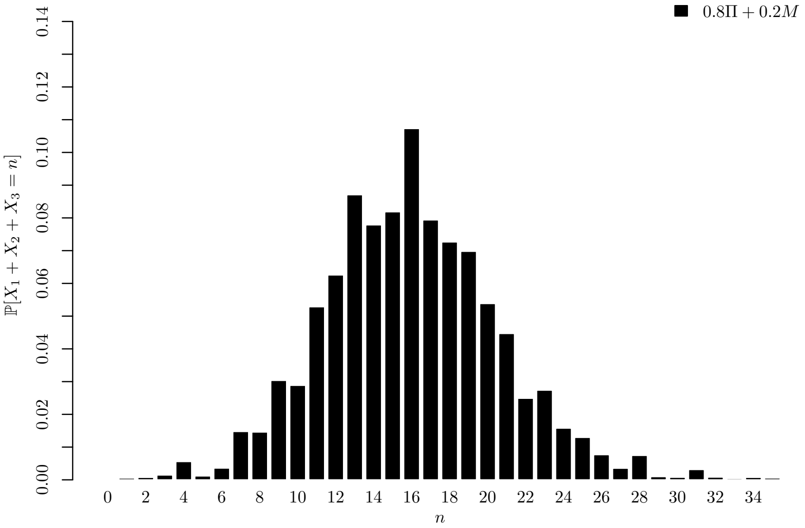

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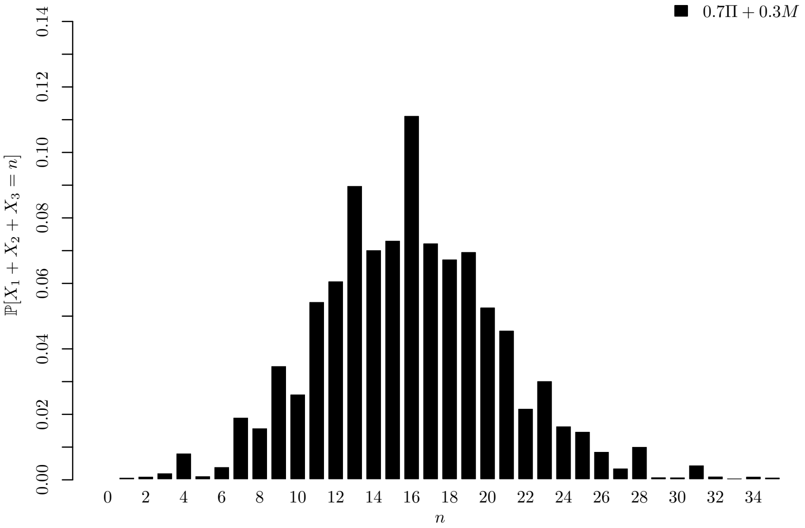

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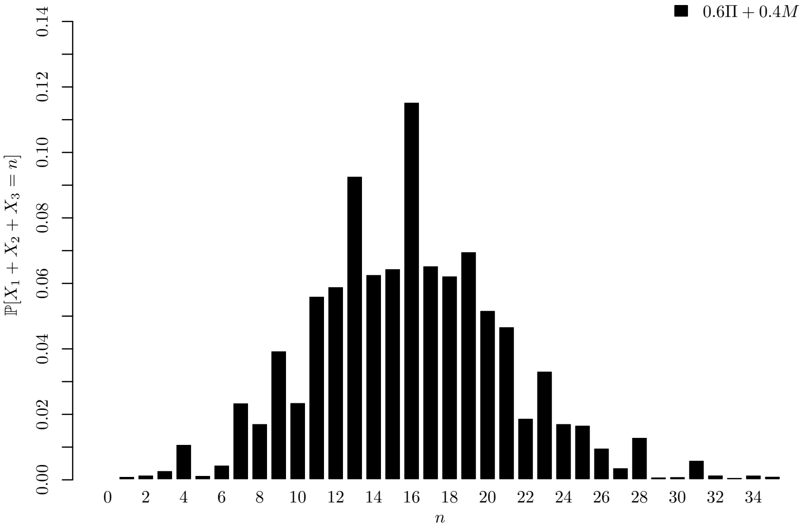

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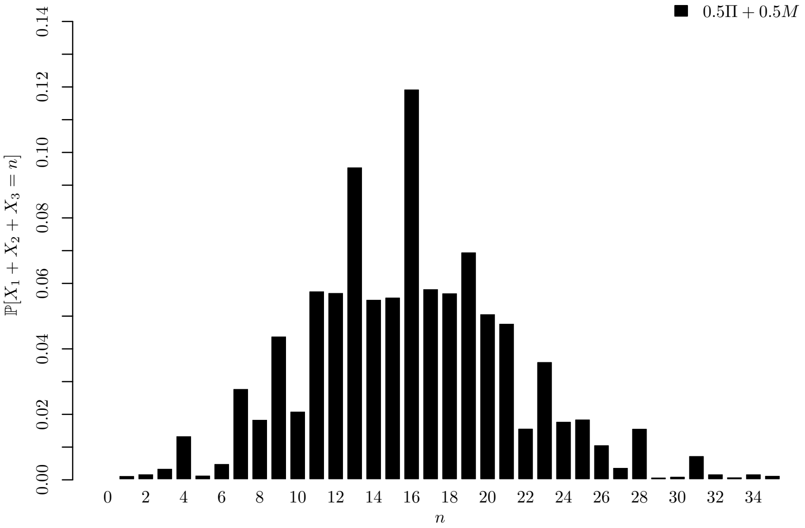

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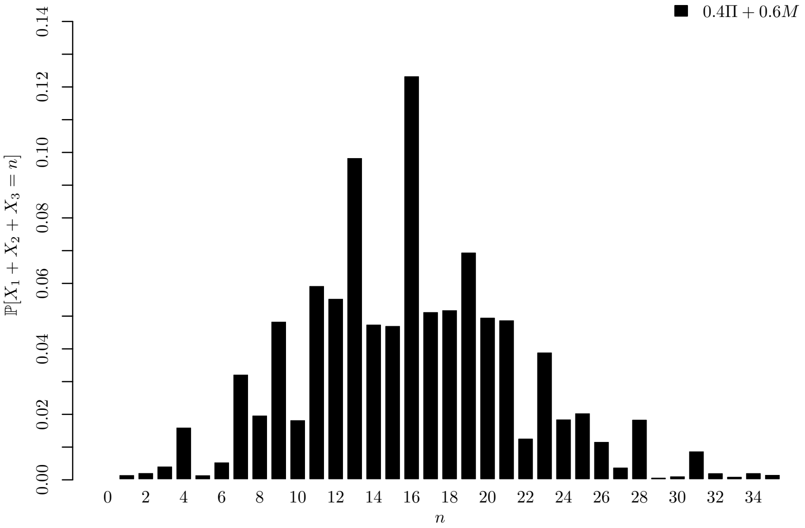

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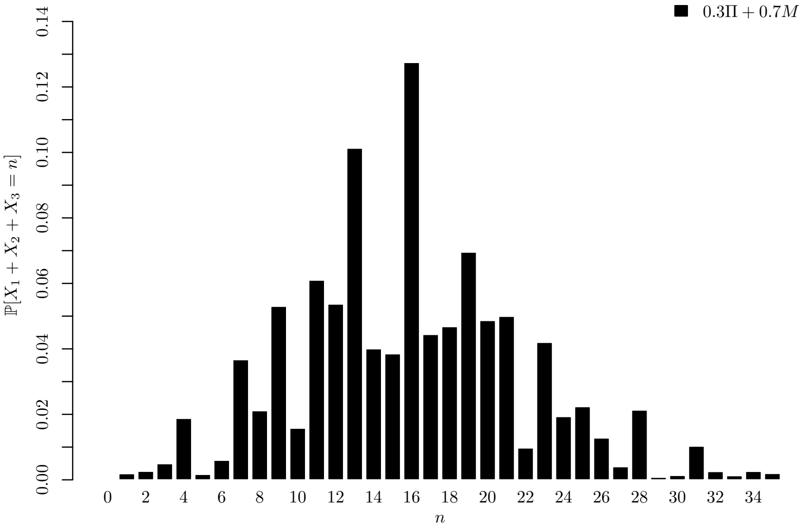

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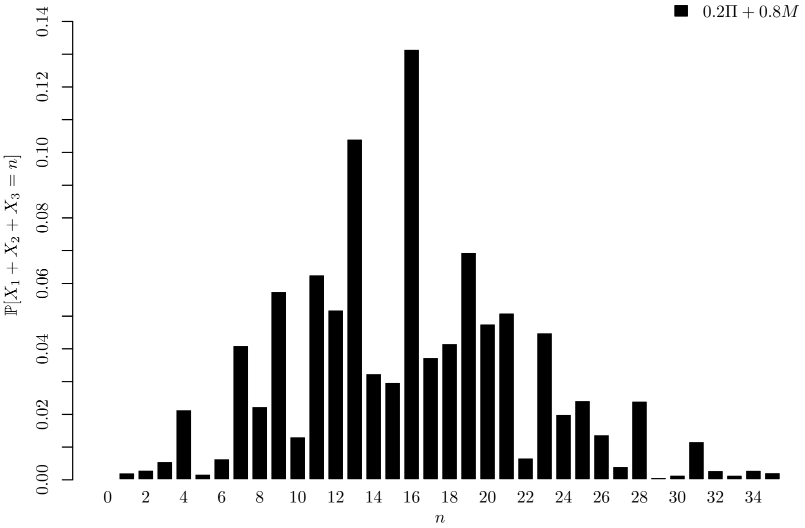

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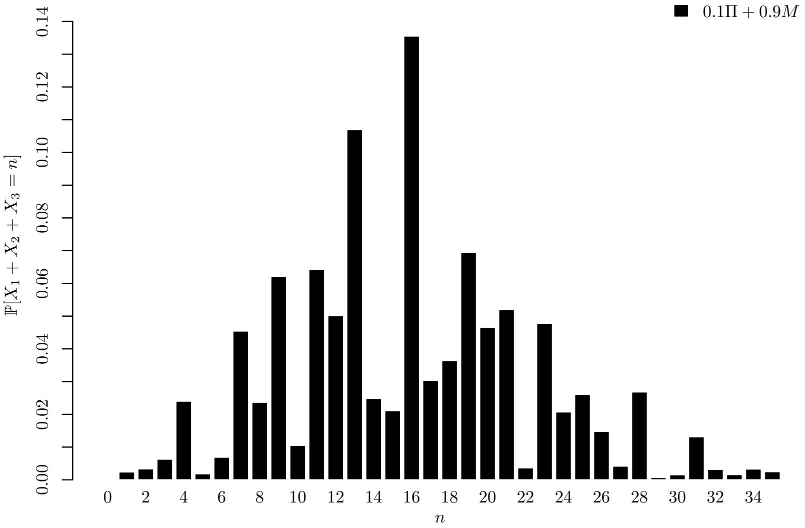

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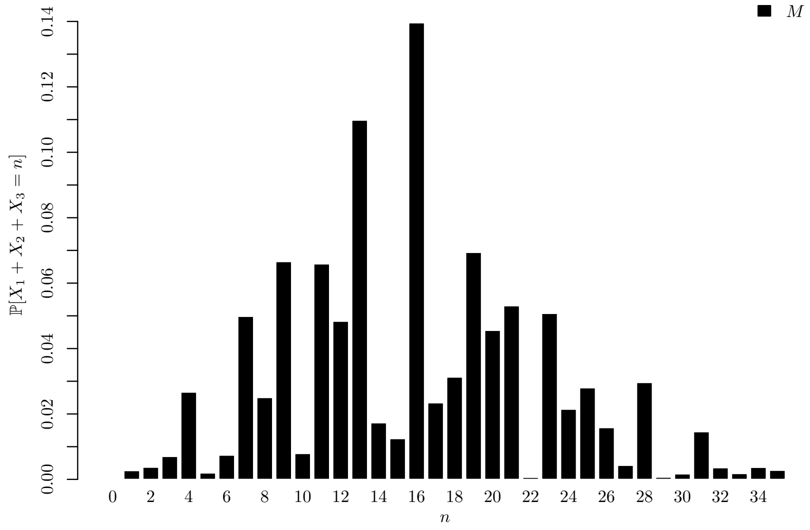

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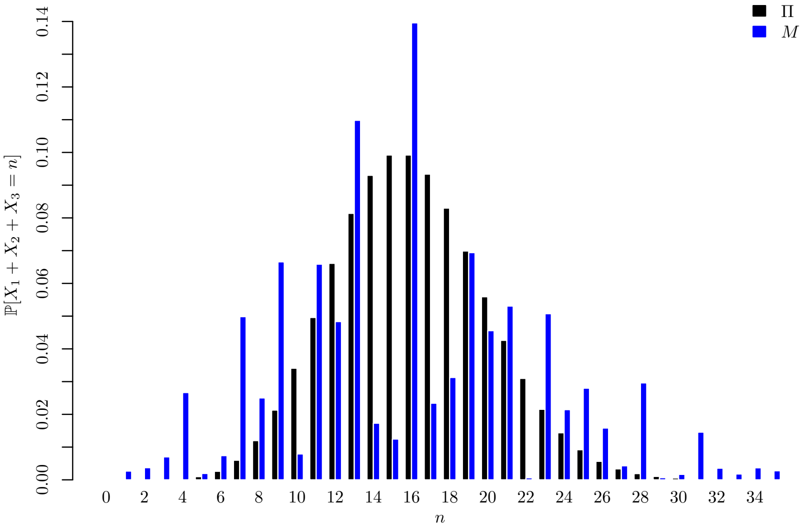


Gaussian copulas

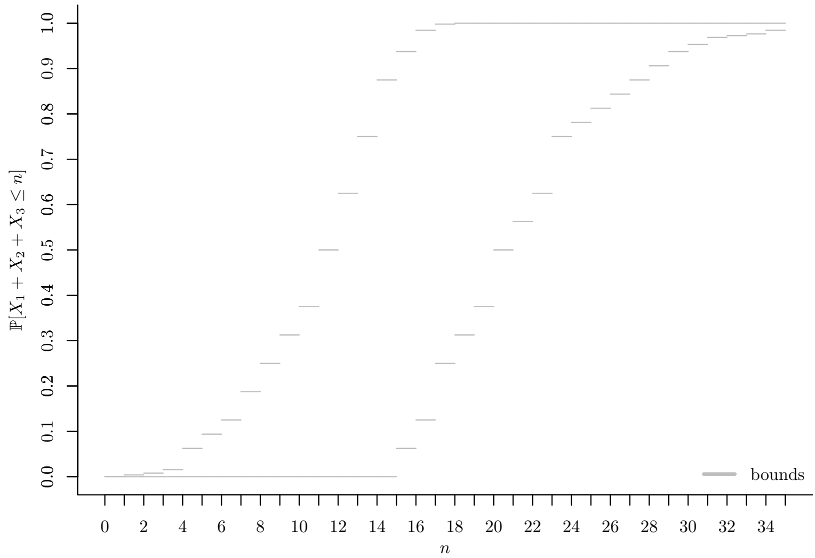
Notes for the following plots:

- By a Gaussian copula with parameter ρ, i.e. C_ρ^{Ga}, we denote a Gaussian copula with correlation matrix such that all pairwise correlations coincide to $\rho \in [-1, 1]$.
- For the Poisson-distributed random variables from our example it holds that $X_1 + X_2 \overset{d}{=} X_3$, if X_1 and X_2 are independent. So if we try to minimize the variance of the sum $X_1 + X_2 + X_3$ under a Gaussian copula we can use

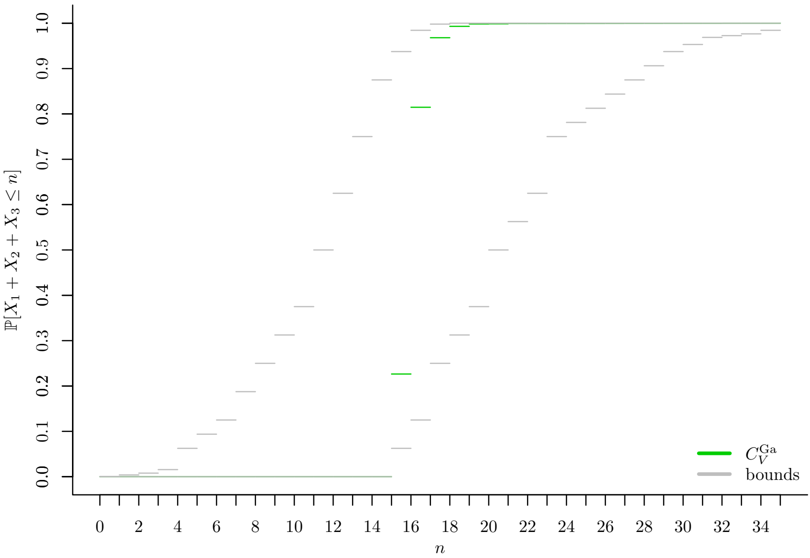
$$V = \begin{pmatrix} 1 & 0 & -\sqrt{\frac{3}{8}} \\ 0 & 1 & -\sqrt{\frac{5}{8}} \\ -\sqrt{\frac{3}{8}} & -\sqrt{\frac{5}{8}} & 1 \end{pmatrix},$$

where the entries of the correlation matrix V are obtained by simple calculation under positive semi-definite constraints.

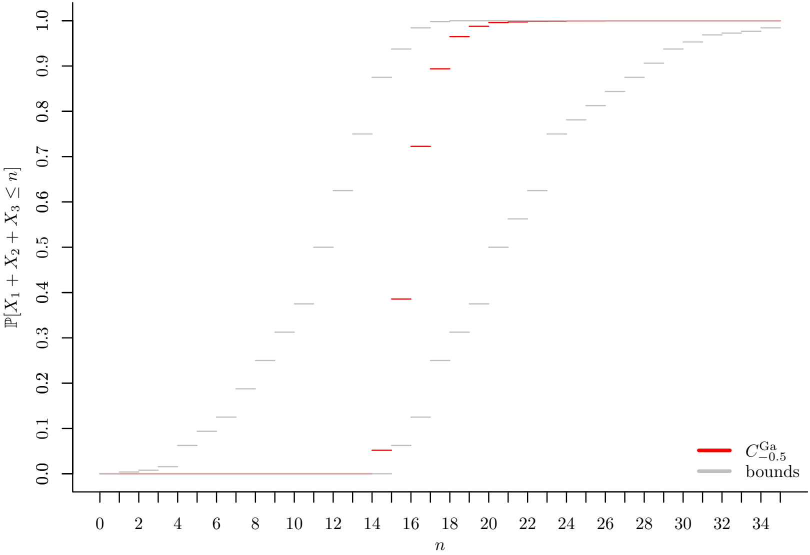
Gaussian copulas



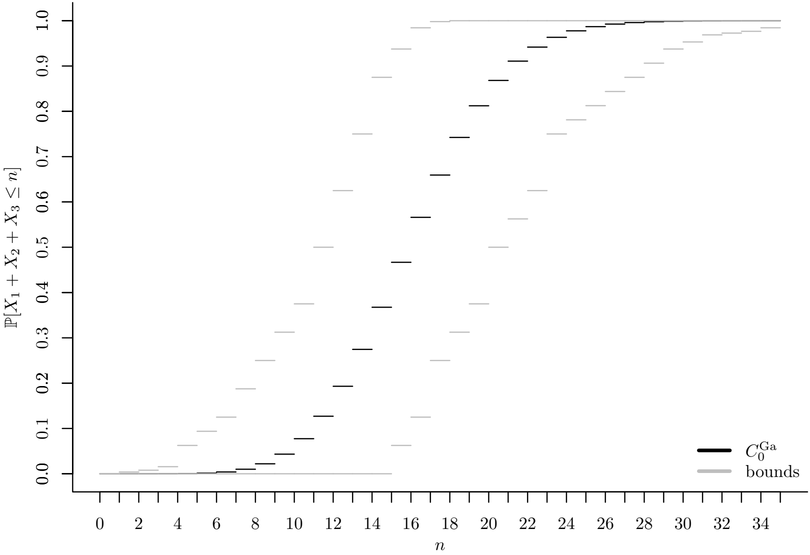
Gaussian copulas



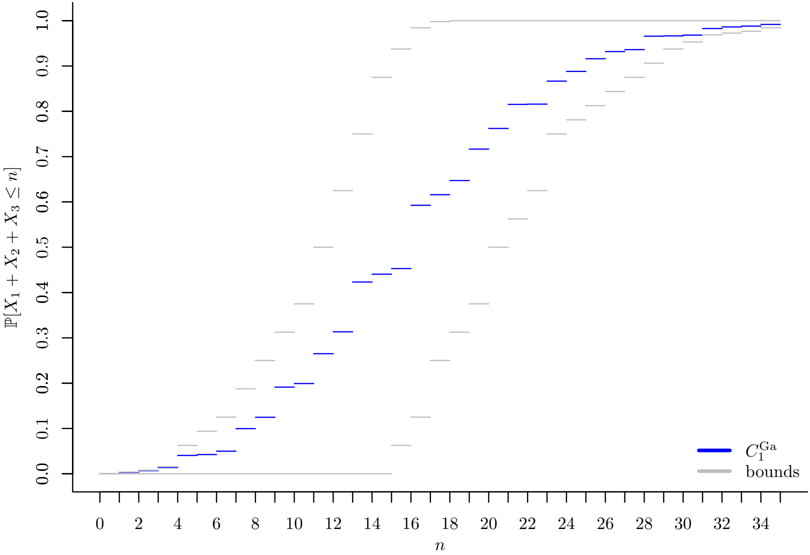
Gaussian copulas



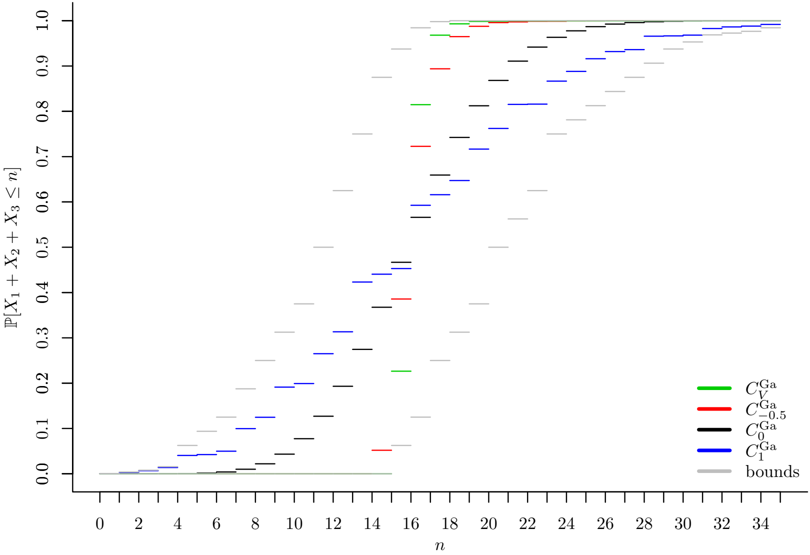
Gaussian copulas



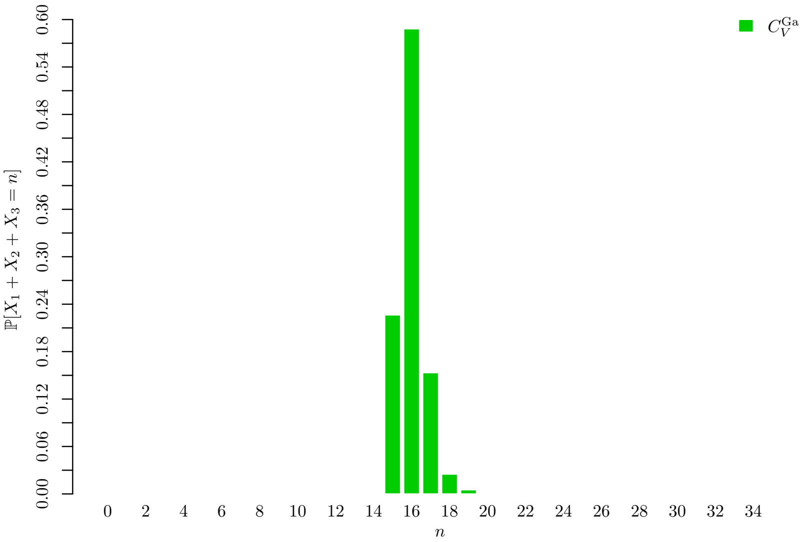
Gaussian copulas



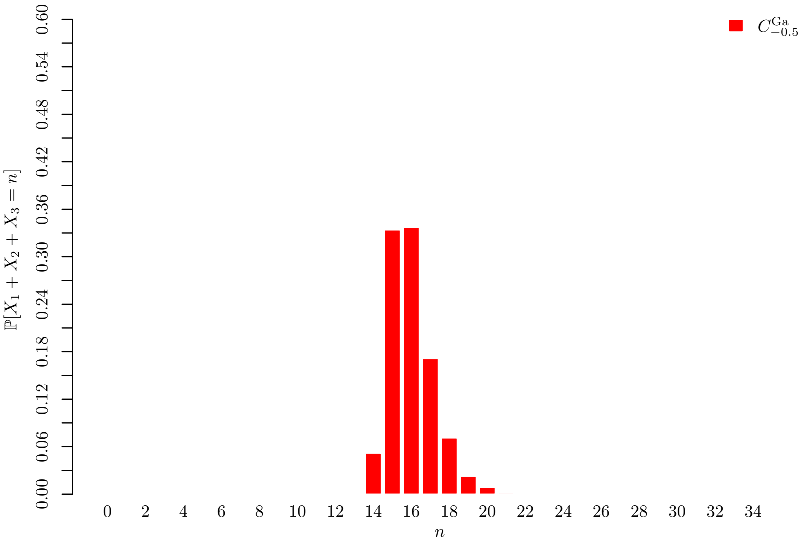
Gaussian copulas



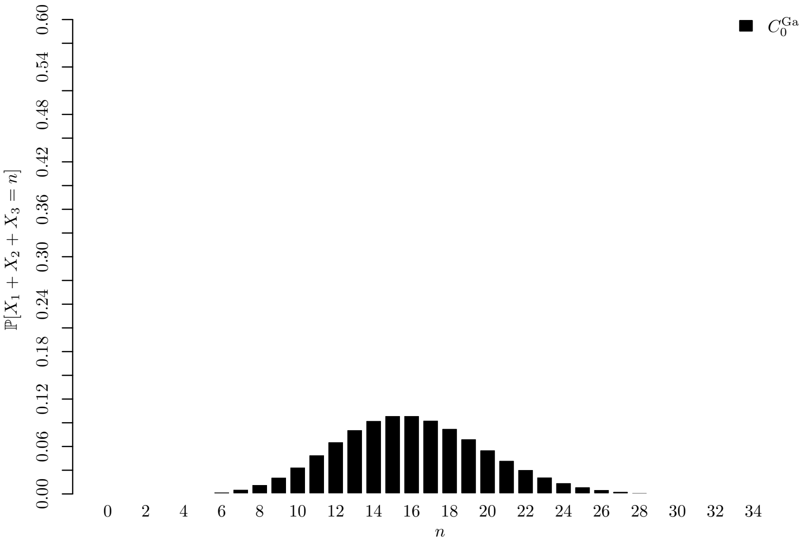
Gaussian copulas



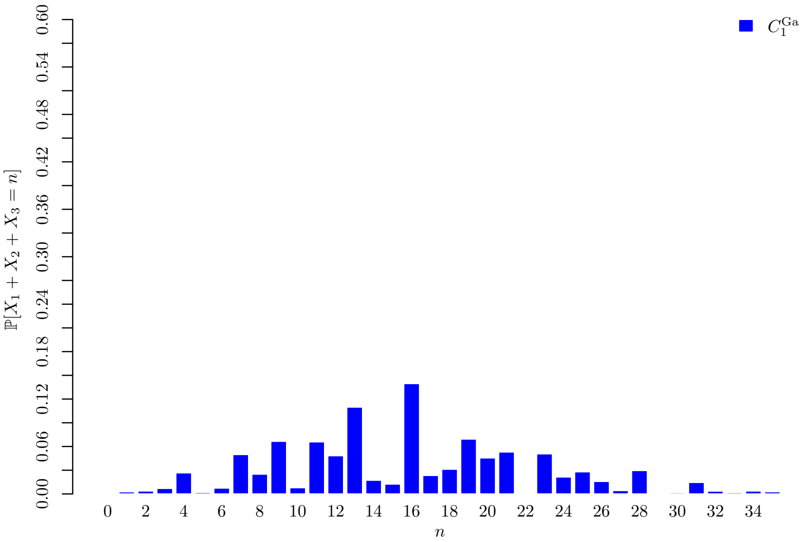
Gaussian copulas



Gaussian copulas



Gaussian copulas



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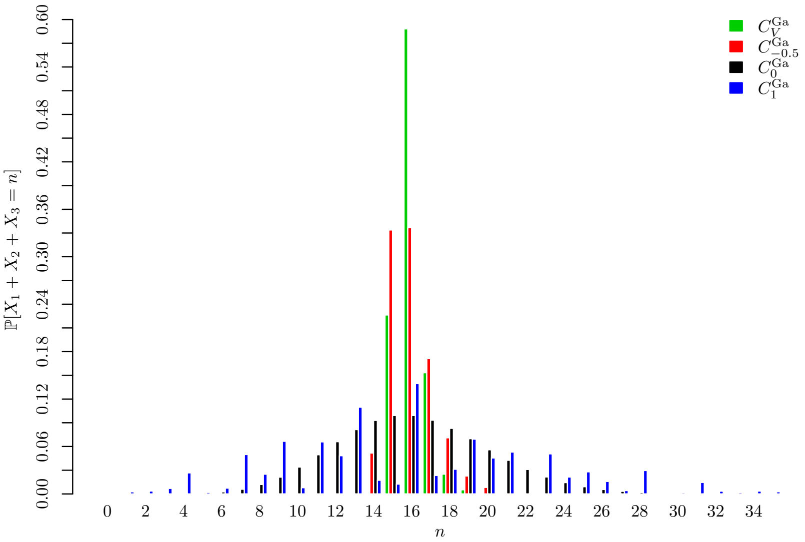
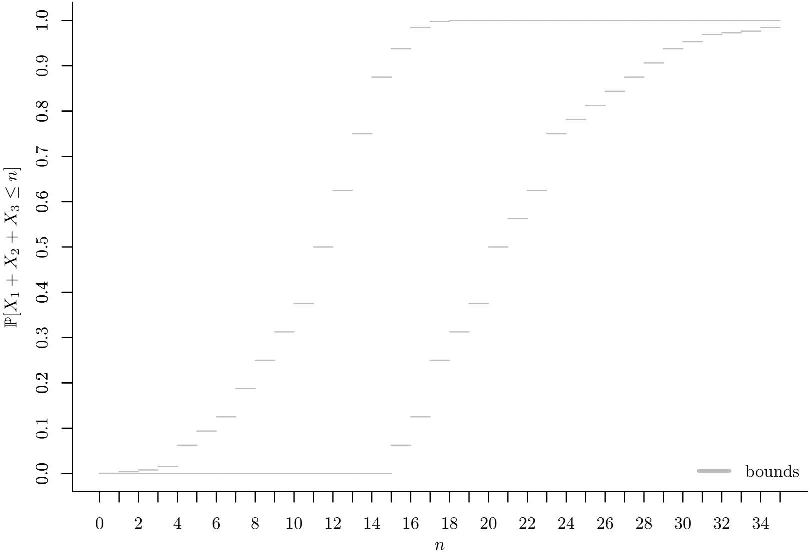


t-copulas

Note for the following plots:

- By a t-copula with parameters ν and ρ, i.e. $C^t_{\nu,\rho}$, we denote a t-copula with ν degrees of freedom and dispersion matrix with 1 in the diagonal and all other entries coinciding to $\rho \in [-1, 1]$.

t-copulas


$\mathbb{P}[X_1 + X_2 + X_3 \leq n]$
0.0
0.1
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0.4
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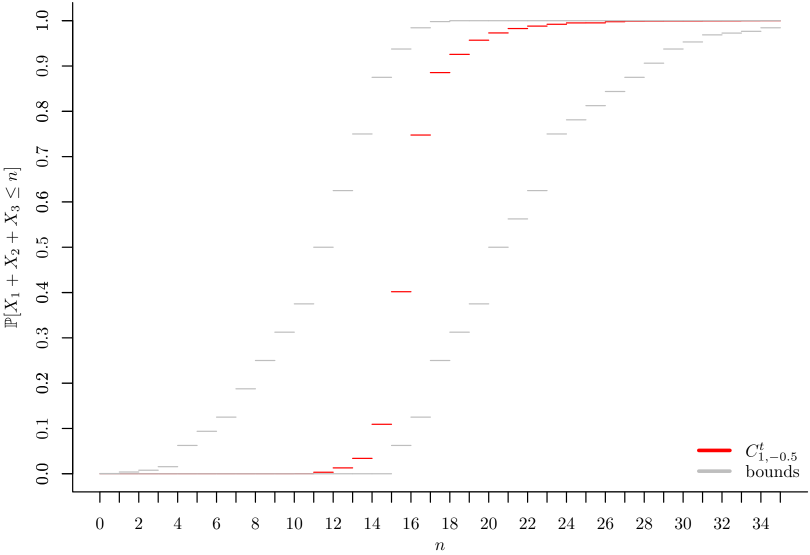
n
bounds

t-copulas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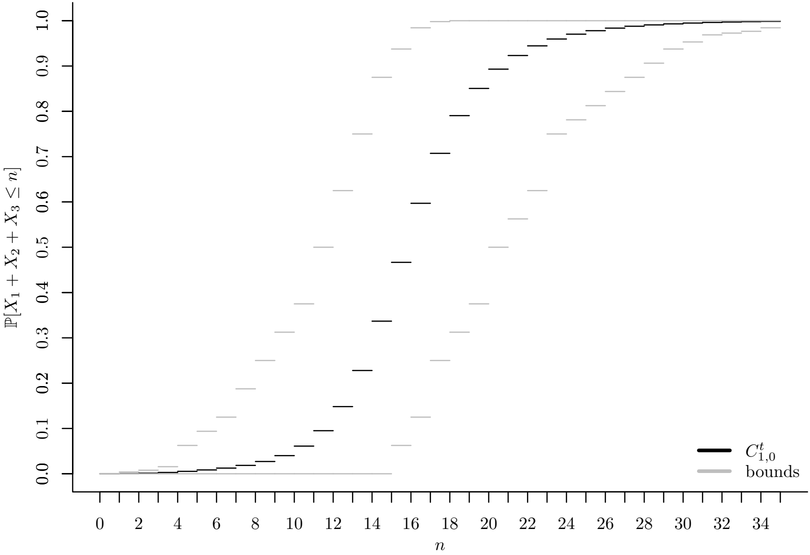
$C^t_{1,-0.5}$
bounds

t-copulas


$\mathbb{P}[X_1 + X_2 + X_3 \leq n]$
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$C^t_{1,0}$
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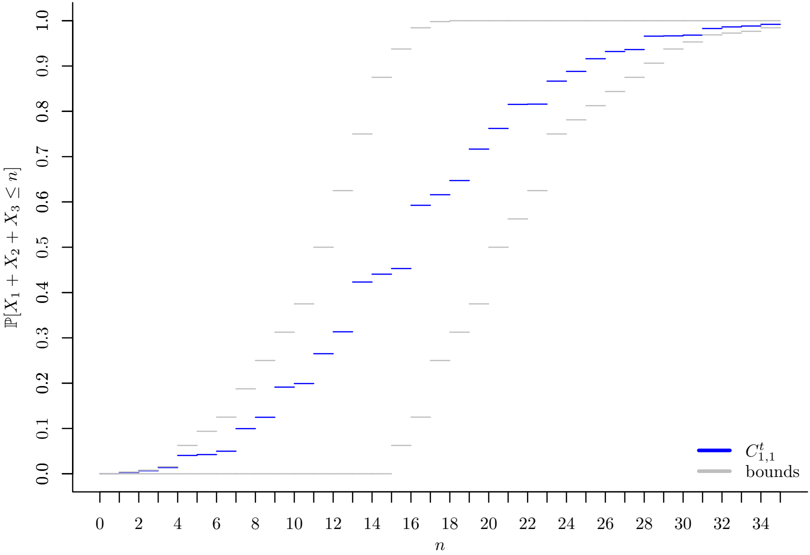
t-copulas


$\mathbb{P}[X_1 + X_2 + X_3 \leq n]$
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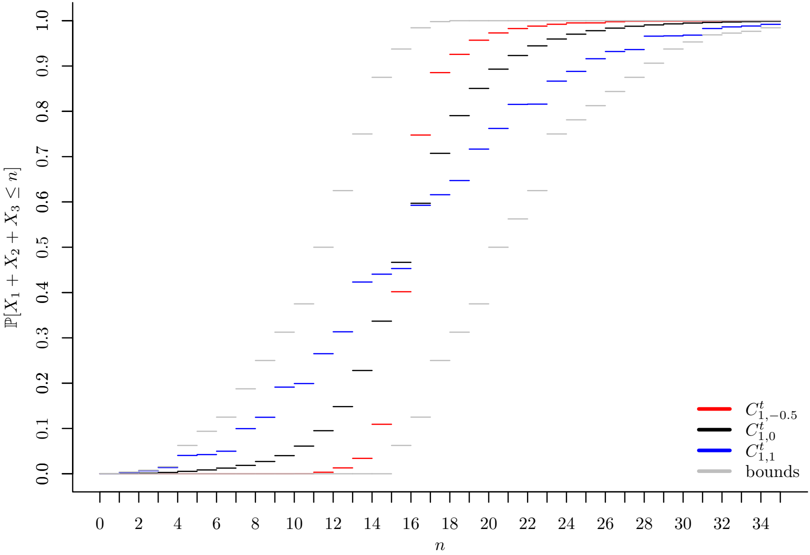
n
$C^t_{1,1}$
bounds

t-copulas


$\mathbb{P}[X_1 + X_2 + X_3 \leq n]$
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0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
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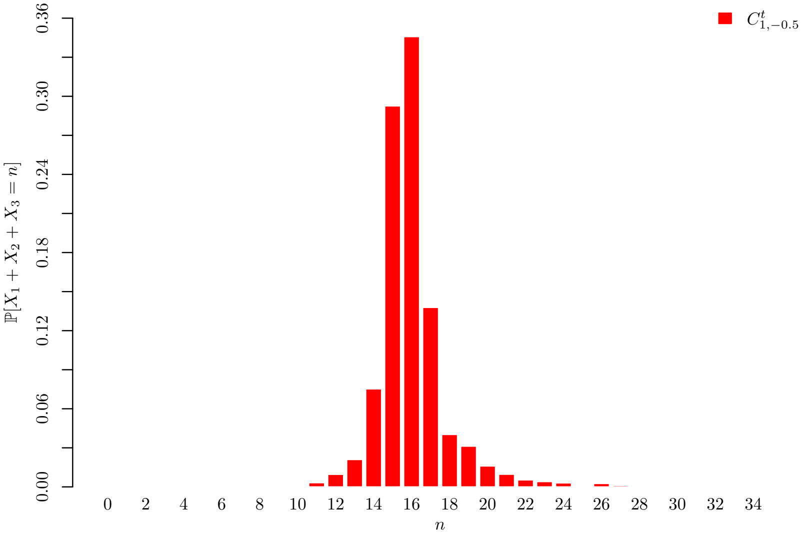
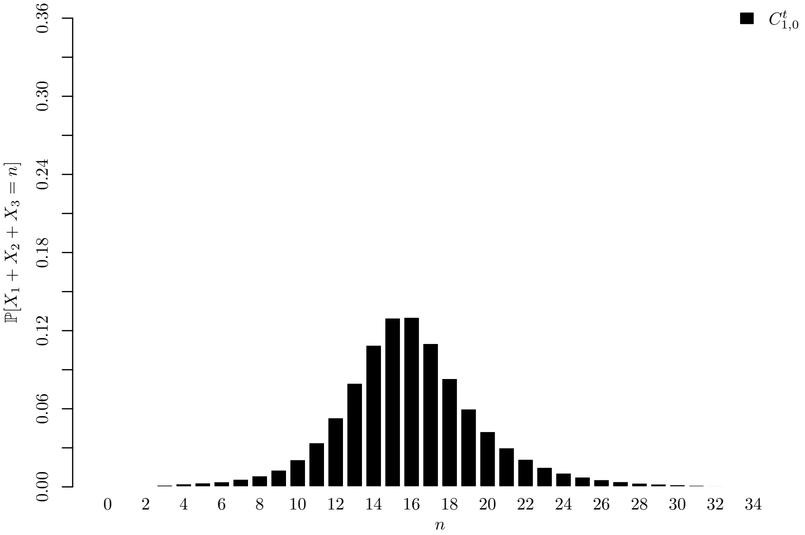
n
$C^t_{1,-0.5}$
$C^t_{1,0}$
$C^t_{1,1}$
bounds

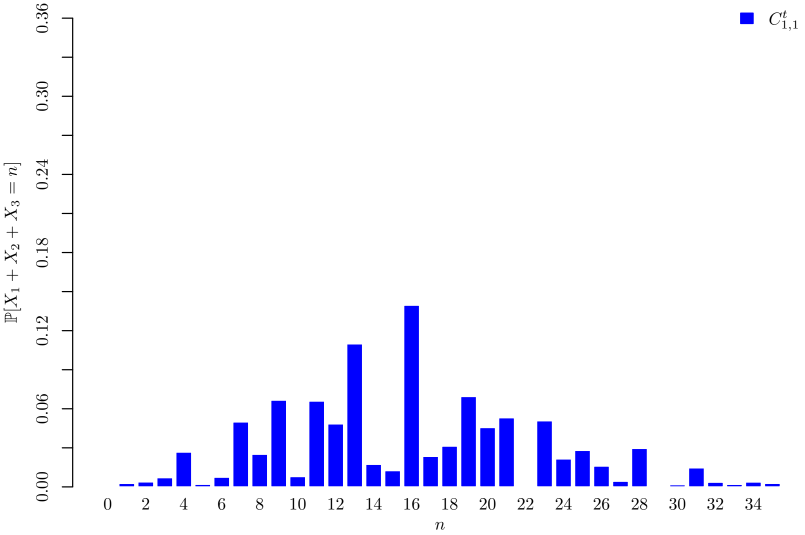
t-copulas


$C^t_{1,-0.5}$
$\mathbb{P}[X_1 + X_2 + X_3 = n]$
0.00
0.06
0.12
0.18
0.24
0.30
0.36
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34
n

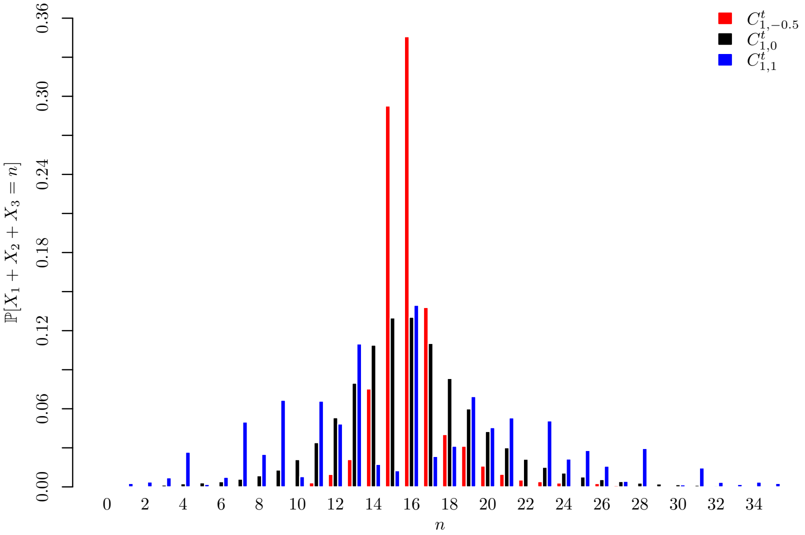
t-copulas


$C^t_{1,0}$
$\mathbb{P}[X_1 + X_2 + X_3 = n]$
0.00
0.06
0.12
0.18
0.24
0.30
0.36
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
n

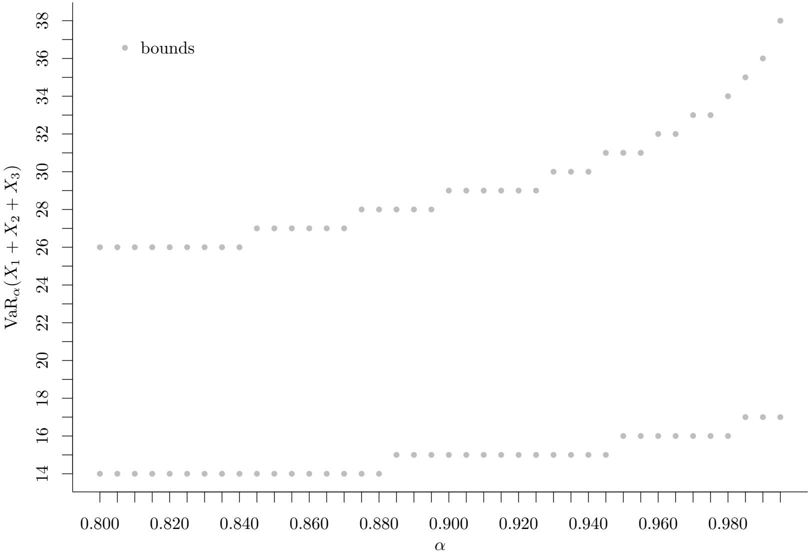
t-copulas


$C^t_{1,1}$
$\mathbb{P}[X_1 + X_2 + X_3 = n]$
0.00
0.06
0.12
0.18
0.24
0.30
0.36
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
n

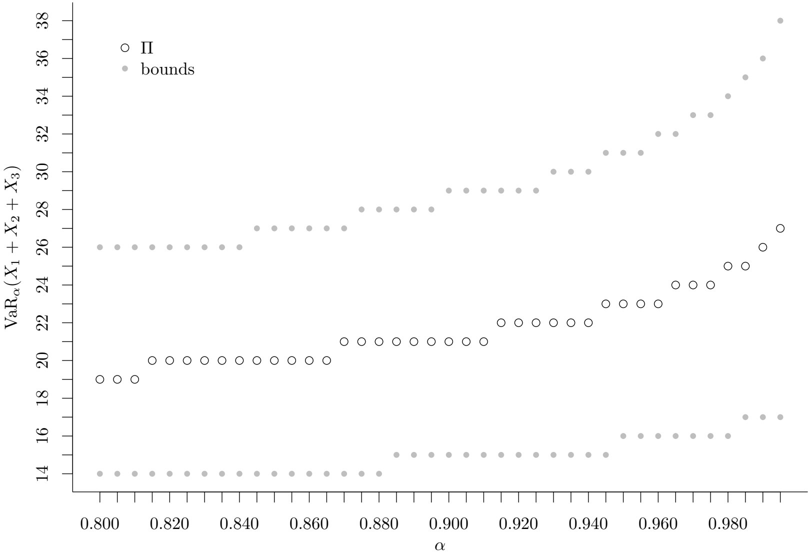
t-copulas


$C^t_{1,-0.5}$
$C^t_{1,0}$
$C^t_{1,1}$
$\mathbb{P}[X_1 + X_2 + X_3 = n]$
0.00
0.06
0.12
0.18
0.24
0.30
0.36
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
n

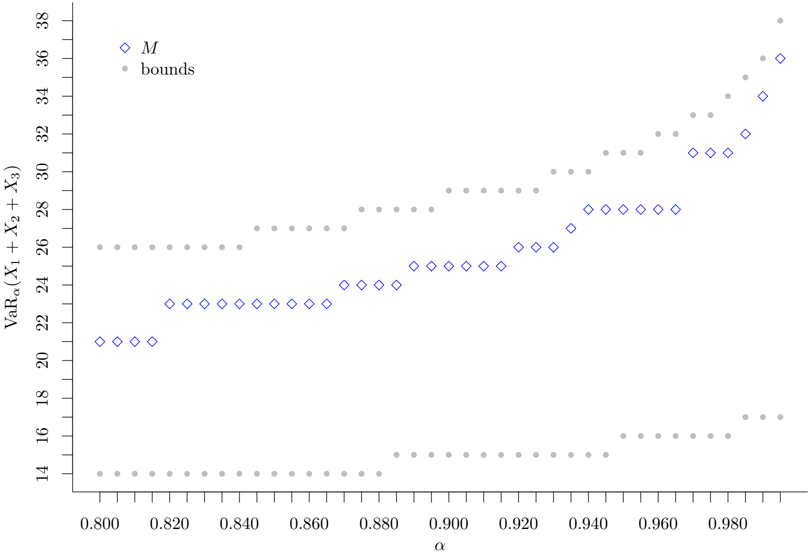
Fundamental copulas – Value-at-Risk



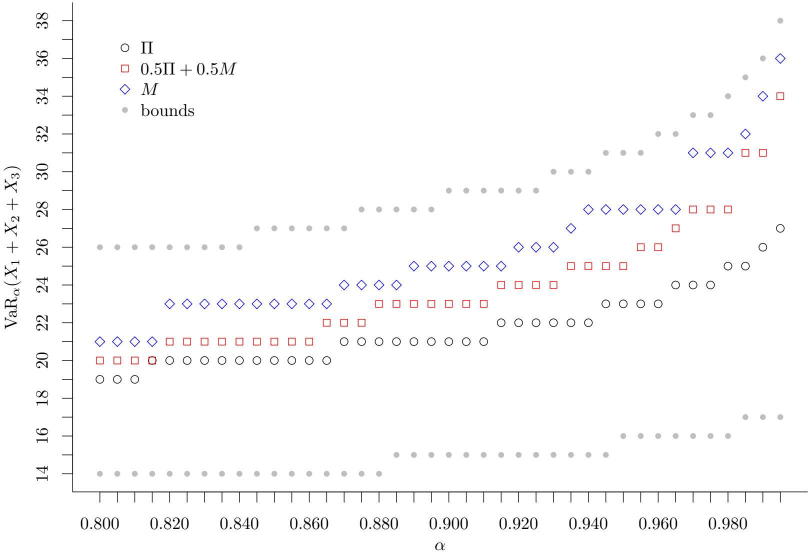
Fundamental copulas – Value-at-Risk



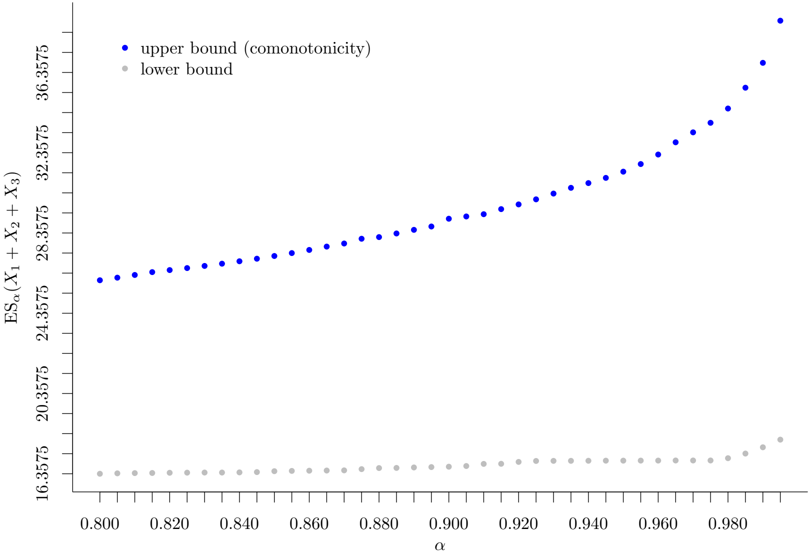
Fundamental copulas – Value-at-Risk



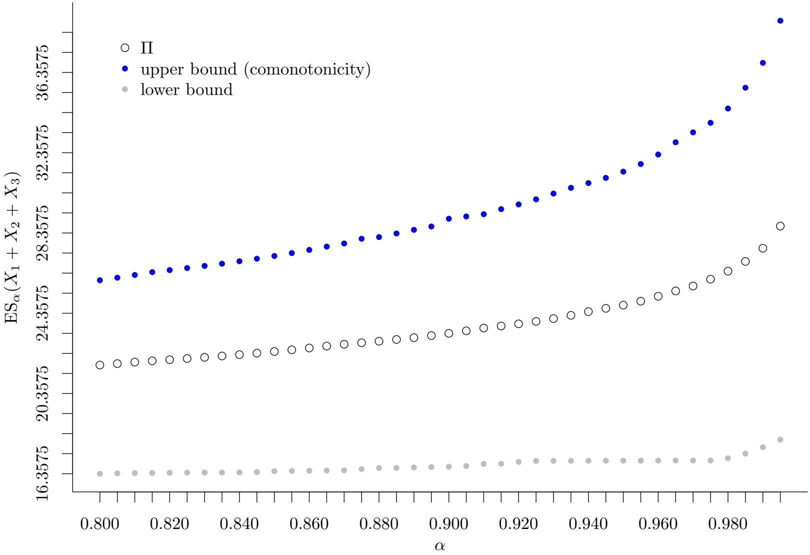
Fundamental copulas – Value-at-Risk



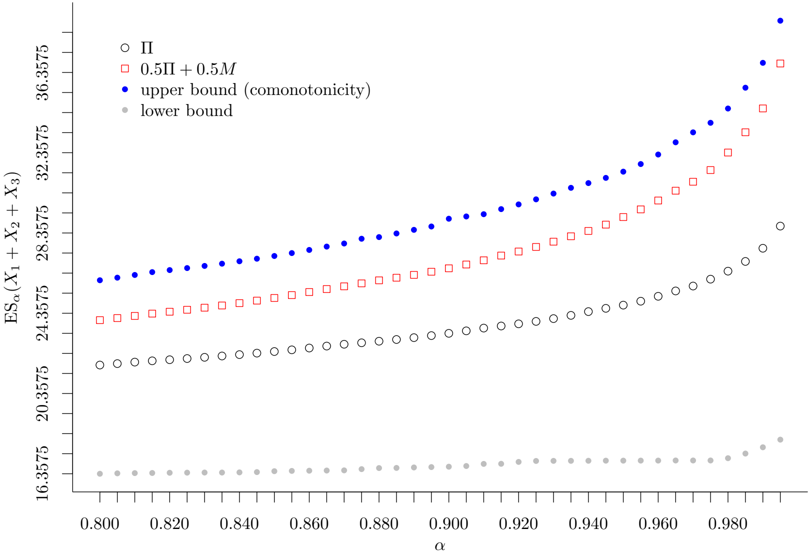
Fundamental copulas – Expected Shortfall



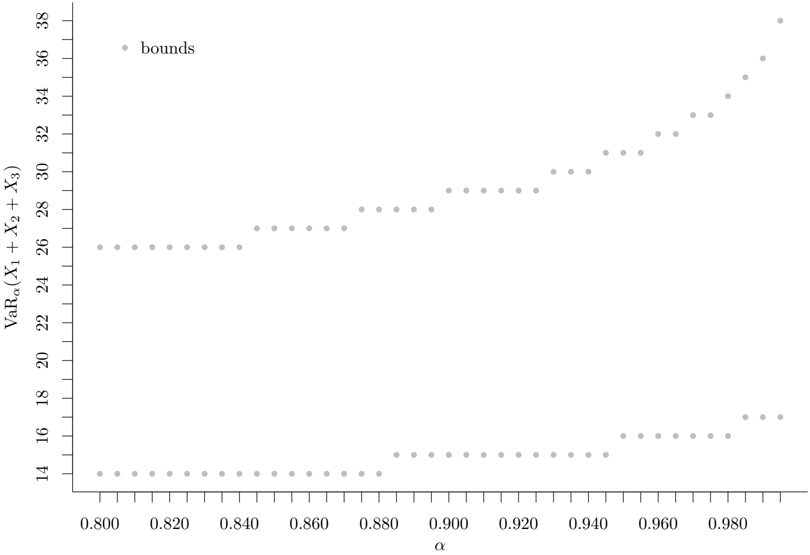
Fundamental copulas – Expected Shortfall



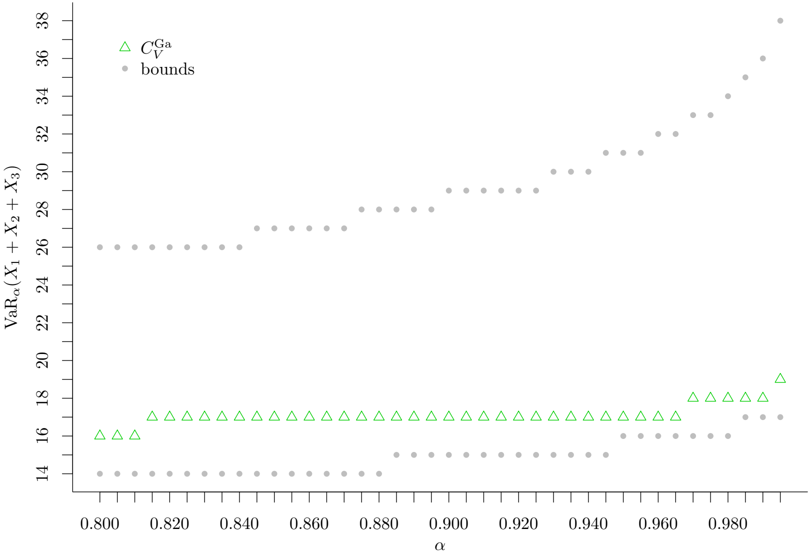
Fundamental copulas – Expected Shortfall



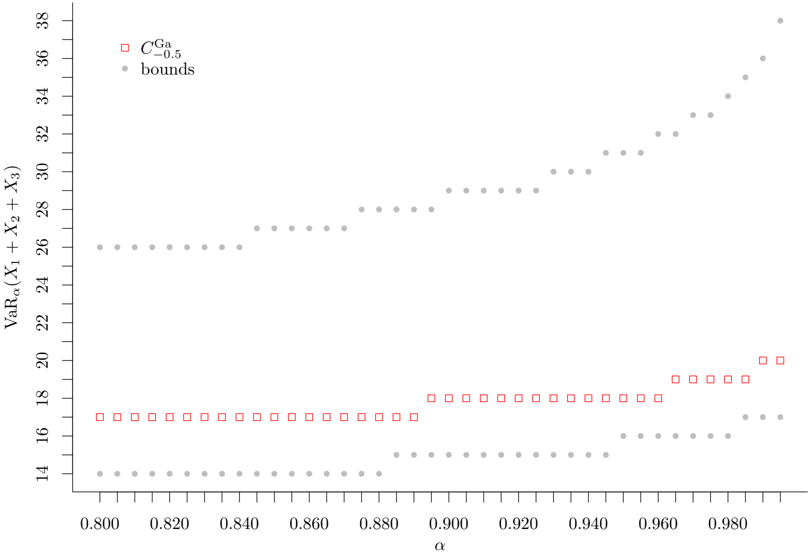
Gaussian copulas – Value-at-Risk



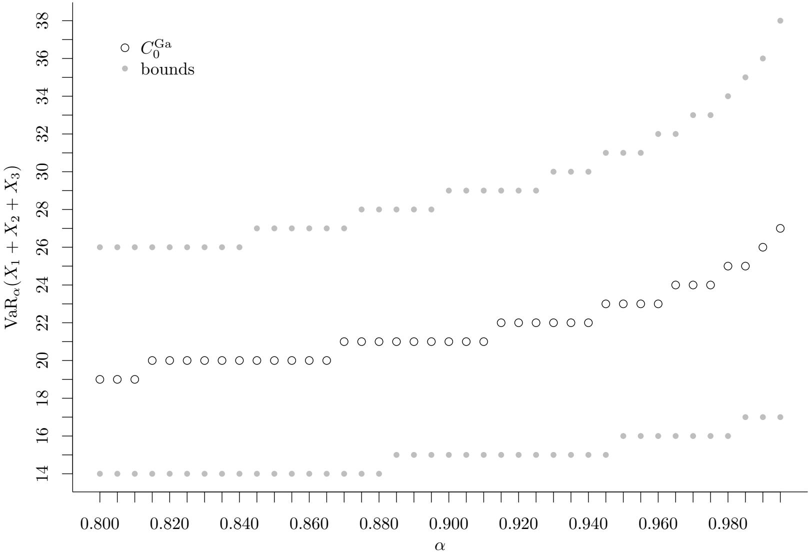
Gaussian copulas – Value-at-Risk



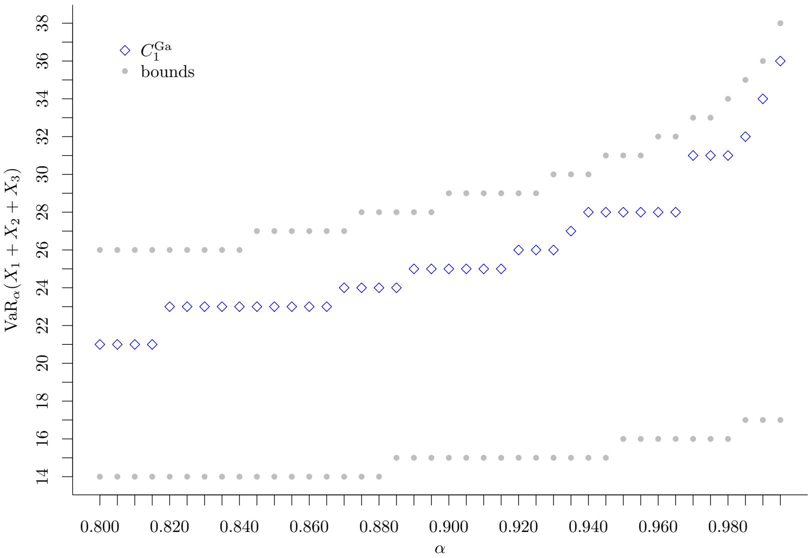
Gaussian copulas – Value-at-Risk



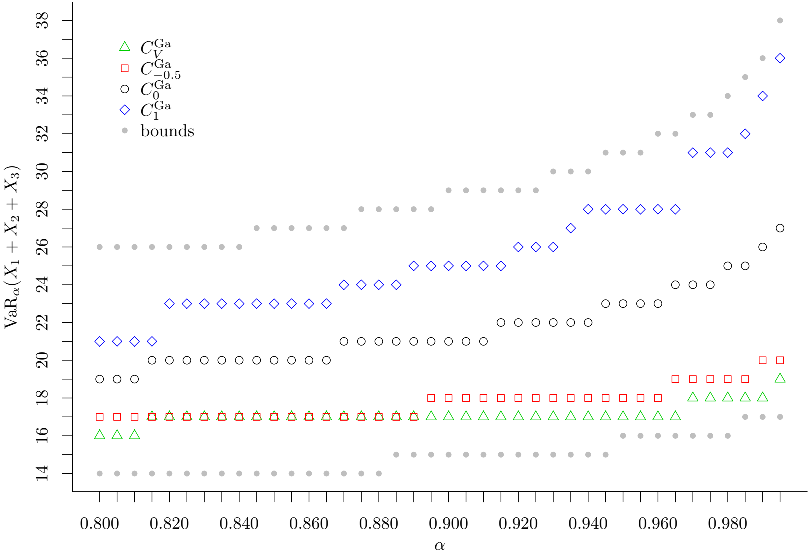
Gaussian copulas – Value-at-Risk



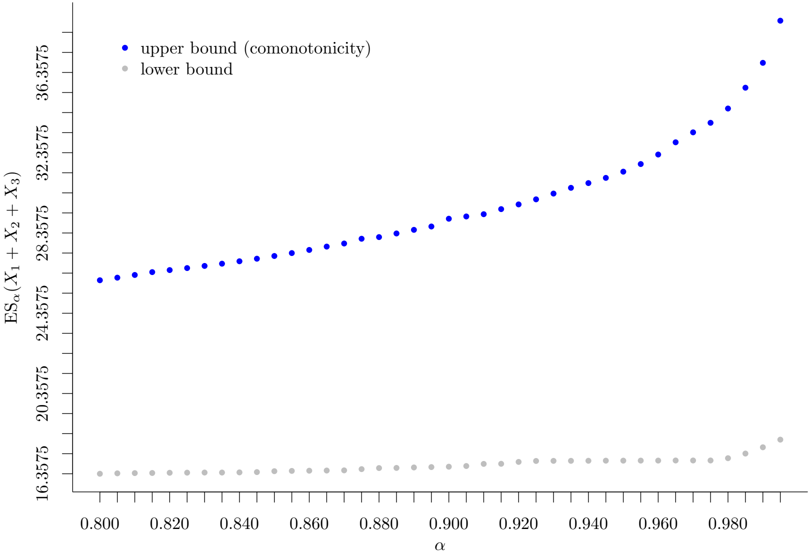
Gaussian copulas – Value-at-Risk



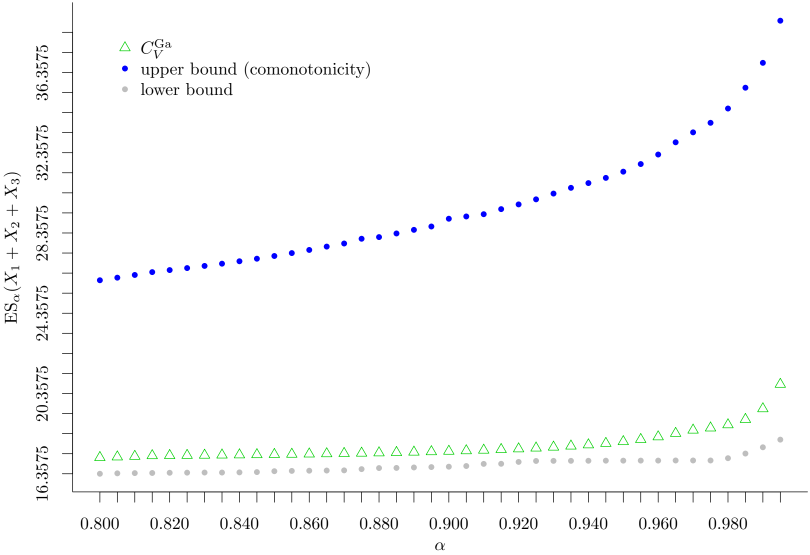
Gaussian copulas – Value-at-Risk



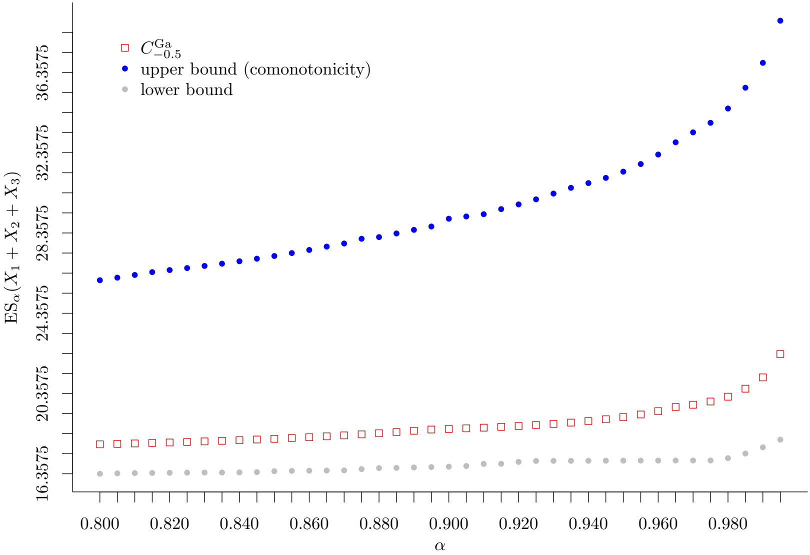
Gaussian copulas – Expected Shortfall



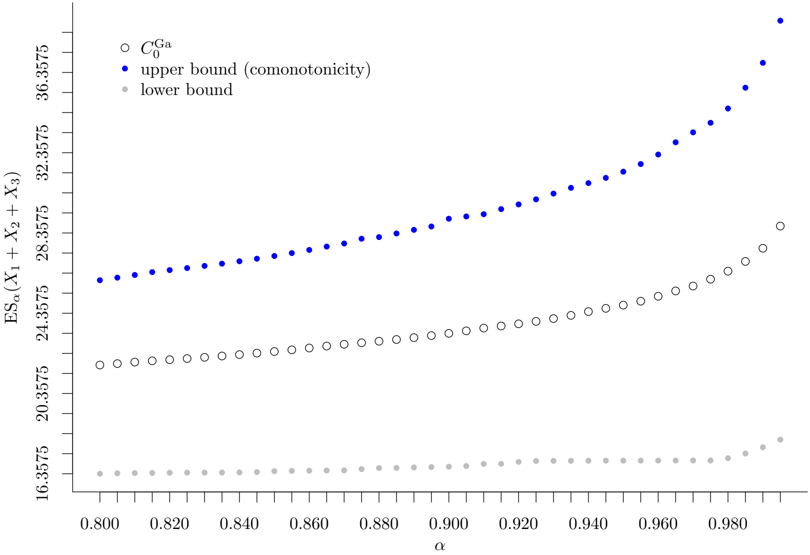
Gaussian copulas – Expected Shortfall



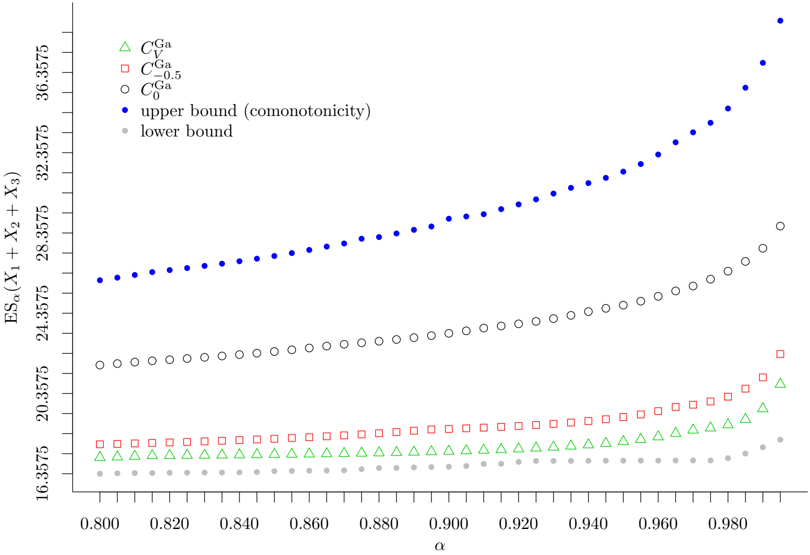
Gaussian copulas – Expected Shortfall



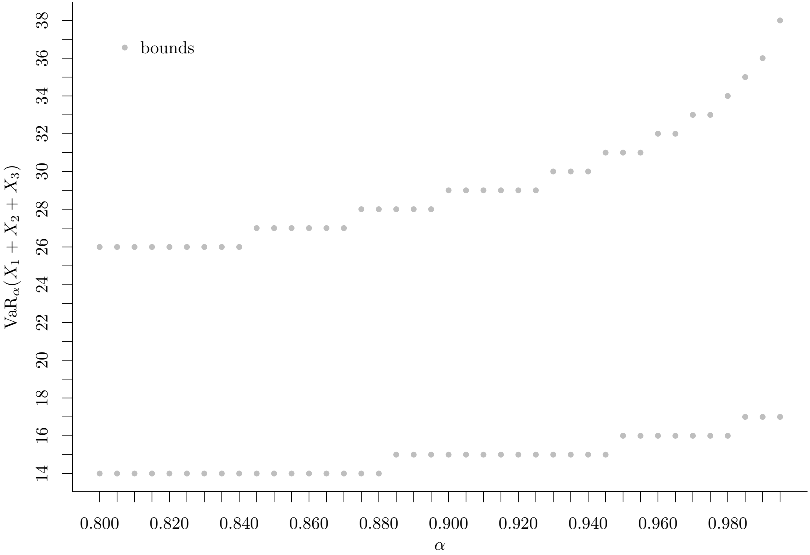
Gaussian copulas – Expected Shortfall



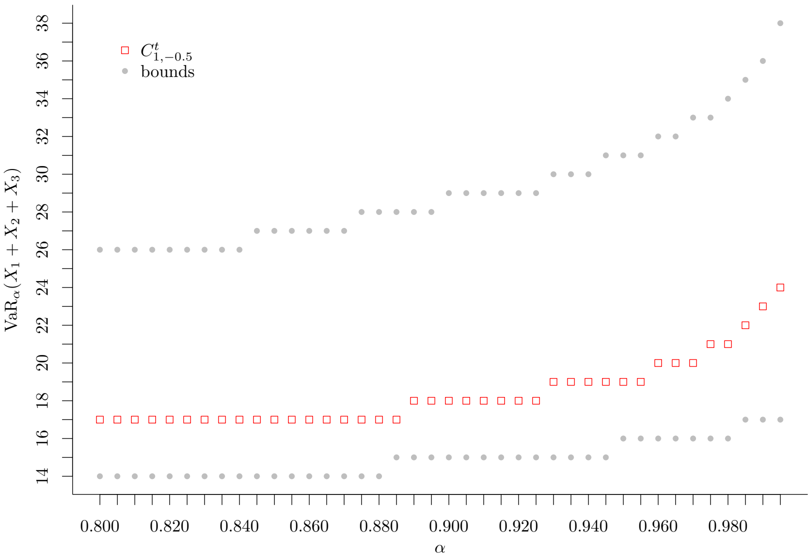
Gaussian copulas – Expected Shortfall



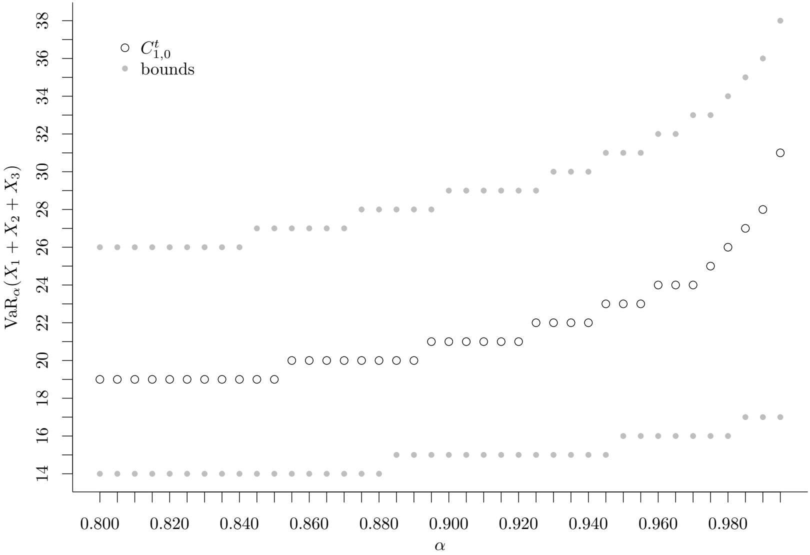
t-copulas – Value-at-Risk



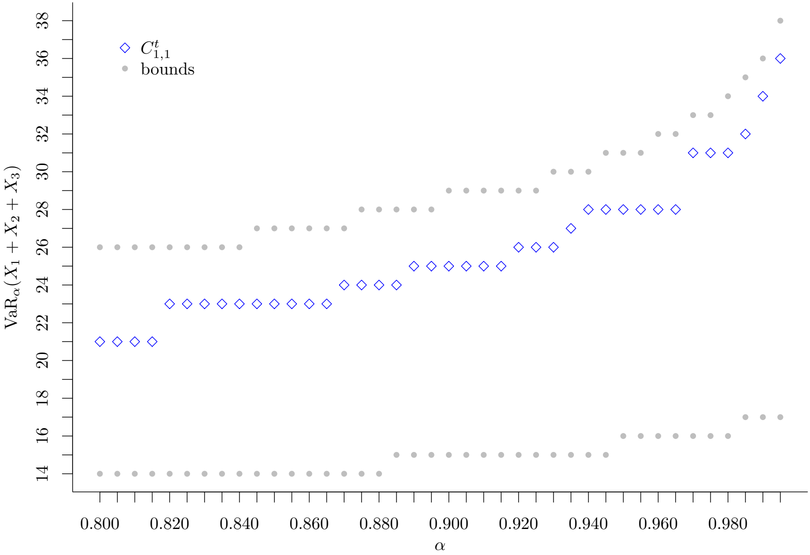
t-copulas – Value-at-Risk



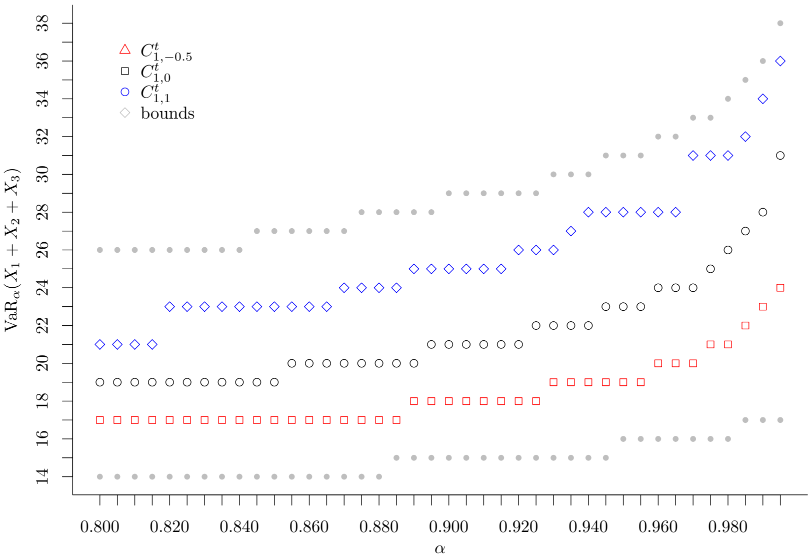
t-copulas – Value-at-Risk



t-copulas – Value-at-Risk



t-copulas – Value-at-Risk



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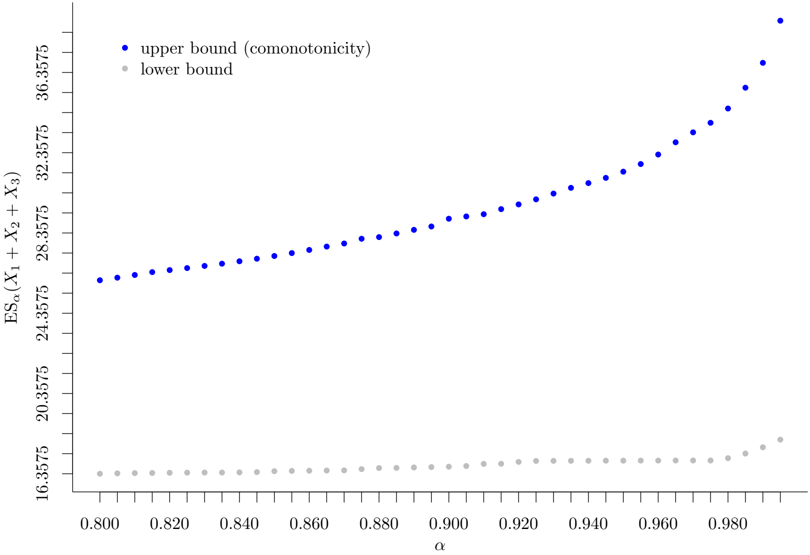

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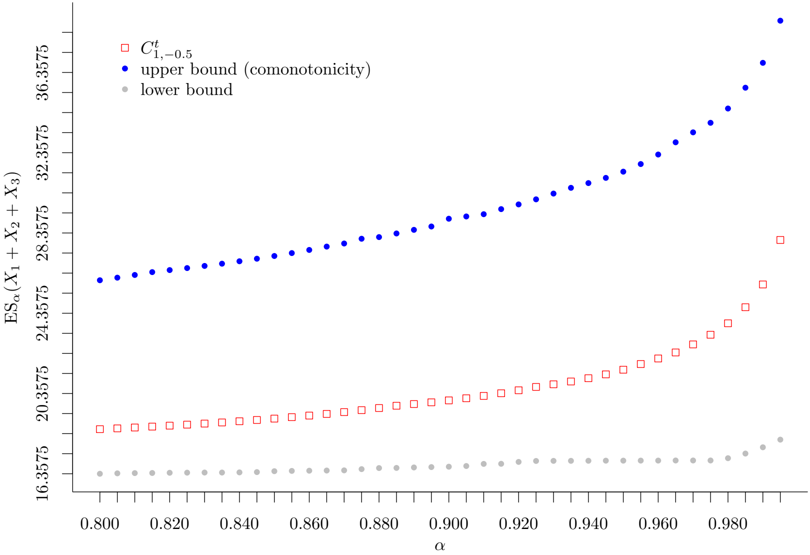

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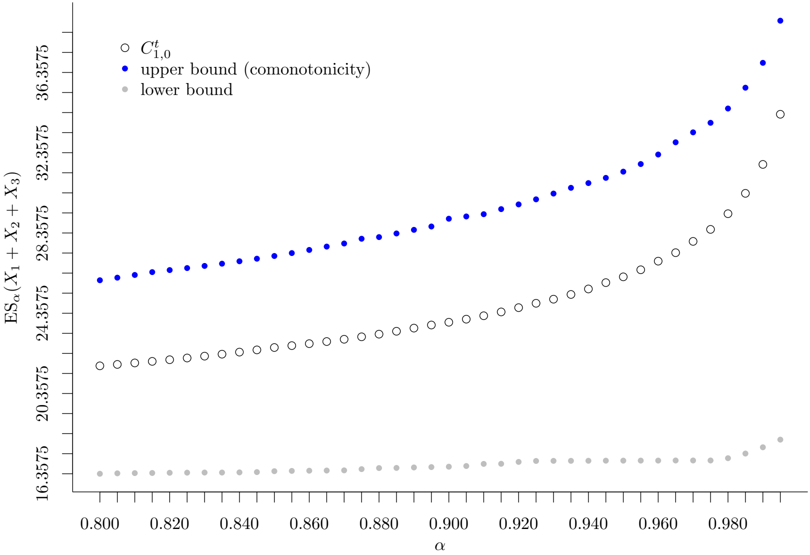

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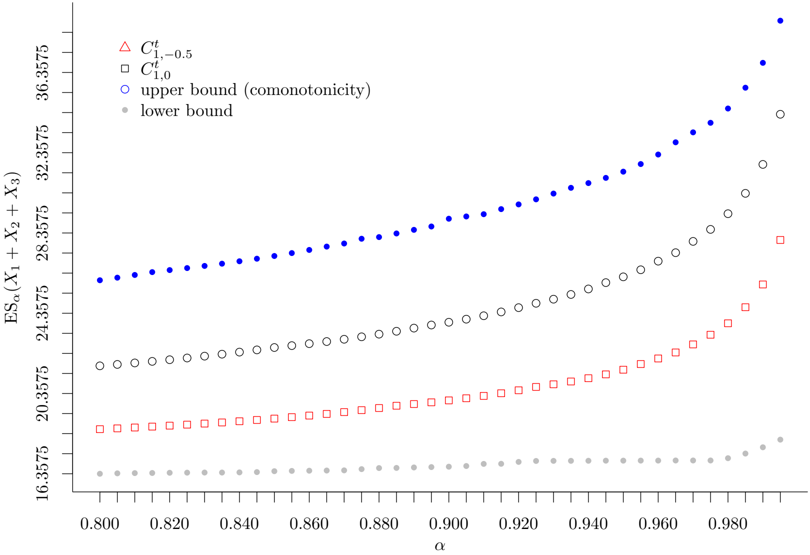

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